On trees invariant under edge contraction

Pascal Maillard (Université Paris-Sud)

based on joint work with

Olivier Hénard (Université Paris-Sud)

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Problem statement (1)

$T = (V, E, \rho)$ random rooted tree (in the graph theoretic sense), locally finite. For $p \in (0, 1)$, define the random tree $\mathcal{C}_p(T)$ by *contracting* each edge in T with probability $1 - p$. Contracting an edge means removing it and identifying its head and tail.

Equivalent definition: $V' =$ set containing each vertex with probability p (plus root). Construct tree on V' by preserving ancestral relationships.

Note: Resulting tree need not be locally finite (if the critical point p_c of edge percolation on the tree satisfies $p_c < 1 - p$)

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Characterize/construct all p-self-similar trees.

Related works

Large body of literature concerning dynamics on random trees:

- Growth (Rémy (1985), Aldous (1991), Duquesne and Winkel (2007)...)
- Percolation on leaves (Aldous and Pitman (1998),...)
- Subtree pruning and regrafting (Evans and Winter (2006),...)
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But here for us more relevant: Janson (2011): exchangeable random partially ordered sets.

Trivialities

Problem

Characterize/construct all p-self-similar trees.

Necessary conditions for T to be self-similar:

- T is infinite
- Finite number of infinite rays, separating at root.

Trivial examples of p-self-similar trees: $\mathbb{N}$, $\mathbb{N} \sqcup \ldots \sqcup \mathbb{N}$.

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Less trivial example

$\mathbb{N}$, attach to each vertex bouquets of edges, numbers are iid geometrically distributed

Main result (informal statement)

Theorem S

Any p-self-similar tree T can be obtained by **Poissonian sampling** from a **real, rooted, measured, random tree**, which itself satisfies a certain natural **scale invariance** property. Conversely, every such real tree defines a p-self-similar tree T through Poissonian sampling.

The real tree in the above theorem can be seen as a certain *scaling limit* of the discrete tree T.

WARNING!

Some notation follows...

A convention

For a metric space X, define $\mathcal{M}_1(X)$ the space of probability measures on X, endowed with Prokhorov's topology. In what follows, we will often study operations on laws of random variables (such as the law of a random tree). We will often identify a random variable with its law and write for example $T \in \mathcal{M}_1(\mathbb{T})$, for $\mathbb{T}$ the space of locally finite rooted trees.

We also use without mention that a continuous map $f : X \to Y$ or $f : X \to \mathcal{M}_1(Y)$ can be *canonically extended* to a continuous map $f : \mathcal{M}_1(X) \to \mathcal{M}_1(Y)$.

Real trees

A *real tree* is a geodesic metric space $(\mathcal{V}, d)$ "without cycles". There is a natural definition of length/Lebesgue measure $\ell_{\mathcal{T}}$.

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Definition

- $\mathfrak{T}$: space of (equivalence classes of) *measured, rooted, real, locally compact trees* $\mathcal{T} = (\mathcal{V}, d, \rho, \mu)$ where μ is a locally finite measure,
- $\mathfrak{T}_e \subset \mathfrak{T}$ the subspace of trees with a finite number of ends,
- $\mathfrak{T}_1 \subset \mathfrak{T}$ the subspace where μ is a probability measure,
- $\mathfrak{T}^\ell \subset \mathfrak{T}$, $\mathfrak{T}^\ell_e \subset \mathfrak{T}_e$ and $\mathfrak{T}^\ell_1 \subset \mathfrak{T}_1$ the subspaces where $\mu \geq \ell_{\mathcal{T}}$.

We endow these trees with the *Gromov–Hausdorff–Prokhorov topology*, which makes $\mathfrak{T}$ topologically complete (ADH13).

Note: in particular, $\ell_{\mathcal{T}}$ is Radon/locally finite for $\mathcal{T} \in \mathfrak{T}^\ell$. There are important examples of real trees where this is not the case, e.g. Aldous' *(Brownian) continuum random tree*.

Rescaling and discretization of a real tree

We define two operations on the spaces $\mathfrak{T}^\ell$ and $\mathfrak{T}^\ell_e$, respectively: *rescaling* and *discretization/Poissonian sampling*.

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Rescaling: For $\mathcal{T} = (\mathcal{V}, d, \rho, \mu) \in \mathfrak{T}^\ell$ and $p > 0$, we define the rescaled tree $\mathcal{S}_p(\mathcal{T})$ by

$$\mathcal{S}_p(\mathcal{T}) = (\mathcal{V},\, p \cdot d,\, \rho,\, p \cdot \mu).$$

Definition

We say a (random) tree $\mathcal{T}$ taking values in $\mathfrak{T}^\ell$ is p-self-similar, $p \in (0, 1)$, if $\mathcal{T}$ and $\mathcal{S}_p(\mathcal{T})$ are equal in law (up to measure-preserving isometries fixing the root).

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Discretization: For $\mathcal{T} = (\mathcal{V}, d, \rho, \mu) \in \mathfrak{T}^\ell_e$, we define the discretized tree $\mathcal{D}(\mathcal{T})$ as follows: Sample two random (multi-)sets of vertices $V_0, V_1 \subset \mathcal{V}$ according to independent Poisson processes with intensity $\ell_\mathcal{T}$ and $\mu - \ell_\mathcal{T}$, respectively. Then $\mathcal{D}(\mathcal{T})$ is the discrete tree with the following properties:

- The set of vertices is $V = \{\rho\} \cup V_0 \cup V_1$,
- For two vertices $v, w \in V$,

$$v \preceq_{\mathcal{D}(\mathcal{T})} w \iff v \preceq_\mathcal{T} w \textbf{ and } v \in V_0 \cup \{\rho\}.$$

($v \preceq_\mathcal{T} w$ if v lies on geodesic between ρ and w in $\mathcal{T}$)

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Commutation relation

For every $p \in (0,1)$,

$$\mathcal{D} \circ \mathcal{S}_p = \mathcal{C}_p \circ \mathcal{D}.$$

Main result

Theorem S

There exists a one-to-one correspondence between

- random discrete p-self-similar trees T and
- random real p-self-similar trees $\mathcal{T}$ taking values in $\mathfrak{T}_e^\ell$,

given by

$$T = \mathcal{D}(\mathcal{T}).$$

Examples of p-self-similar real trees

Construction through subordination of a real-valued self-similar process. Ingredients:

1. A random real tree $\mathcal{T}_0$ taking values in $\mathfrak{T}_1^\ell$.
2. A real-valued process $(X(t); t \geq 0)$, which is *increasing*, *pure-jump* and satisfies

$$(pX(t); t \geq 0) \stackrel{\text{law}}{=} (X(pt); t \geq 0).$$

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Construct a p-self-similar real tree as follows:

- Start with an infinite ray (the spine).
- For each jump time t of the process X, take an independent copy $\mathcal{T}_0^{(t)}$ of $\mathcal{T}_0$, and attach its rescaling $S_{X(t)-X(t-)}(\mathcal{T}_0^{(t)})$ to the spine at distance t from the root.

Translation invariant trees

Question

Can one construct examples of one-ended p-self-similar trees $\mathcal{T} = (\mathcal{V}, d, \rho, \mu)$ which are *translation invariant* (in law) along the spine?

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Denote by v_t the spine vertex at distance t from the root and by $\mathcal{V}^{\leq t}$ the subset of vertices which are not descendants of v_t. Define the *mass process* $(X(t); t \geq 0)$ by $X(t) = \mu(\mathcal{V}^{\leq t})$. Then $(X(t); t \geq 0)$ is a real-valued, increasing, stochastic process with stationary increments satisfying,

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Theorem (basically Vervaat (1985))

Let $(X(t); t \geq 0)$ be a process as above. Then, almost surely, for every $t \geq 0$, $X(t) = X(1)t$.

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Corollary

A random, one-ended tree $\mathcal{T}$ taking values in $\mathfrak{T}_e^\ell$, which is translation invariant along the spine, is p-self-similar if and only if

$$\mathcal{T} = (\mathbb{R}_+, d_{\mathrm{Eucl}}, 0, Y \cdot \ell), \quad Y \geq 1 \text{ a random variable.}$$

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Corollary

A random, one-ended discrete tree T, which is translation invariant along the spine, is p-self-similar if and only if there exists a (random) $P \in (0, 1]$, such that each subtree of the spine is a tree of height 1 with a Geo(P) number of edges (independently for each vertex on the spine). ($P = 1/Y$).

A generalization

To get more interesting examples, generalize the contraction and rescaling operations $\mathcal{C}_p$ and $\mathcal{S}_p$: Let $p, q \in (0, 1)$.

- $\mathcal{C}_{p,q}$: Defined as $\mathcal{C}_p$, but vertices on the spine are retained with probability q.
- $\mathcal{S}_{p,q}$: Defined as $\mathcal{S}_p$, but distances on the spine are rescaled by q instead of p.

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Definition

A random (discrete) T is (p,q)*-self-similar* if $T \overset{\text{law}}{=} \mathcal{C}_{p,q}(T)$.

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Theorem S holds with p-self-similar replaced by (p,q)-self-similar.

The iid case

In the translation invariant case, many examples can be constructed when $q > p$. Let us consider the case where the subtrees along the spine are iid. Write the (discrete) tree T as $T = (T^0, T^1, \ldots)$, where T^n is the subtree of the n-th vertex on the spine. We construct a (p, q)-self-similar tree where $T^0, T^1, \ldots$ are iid. The ingredients are the following:

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- $(\mathcal{T}_0^n)_{n\geq 0}$: an iid sequence of trees in $\mathfrak{T}_1^\ell$
- ν: a quasi-stationary distribution with eigenvalue q of the Galton–Watson process $(Z_n; n \geq 0)$ with offspring distribution $p_0 = 1-p$, $p_1 = p$. That is, ν satisfies

$$\forall n \in \mathbb{N} : \mathbb{P}_\nu(Z_n \in \cdot \,|\, Z_n > 0) = \nu \quad \text{and} \quad \mathbb{P}_\nu(Z_1 > 0) = q.$$

Maillard (2015): Characterization of these quasi-stationary distributions.

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- A constant $c \in (0, 1]$.

The iid case (2)

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Construct tree $T = (T^0, T^1, \ldots)$, where $T^0, T^1, \ldots$ are iid according to the following law:

T^0 is the union of a Geo(c)-distributed number of iid trees T', where

$$T' \overset{\text{law}}{=} \mathcal{D}(\mathcal{T}_0, N), \quad N \sim \nu.$$

Here, $\mathcal{D}(\mathcal{T}_0, m)$ is the tree $\mathcal{D}(\mathcal{T}_0)$ cond'ed on having m vertices (plus root).

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"Theorem": This example (basically) covers all cases.

Proof of Theorem S

One direction is obvious: If $\mathcal{T}$ is a p-self-similar random $\mathbb{R}$-tree, then by the commutation relation,

$$\mathcal{C}_p(\mathcal{D}(\mathcal{T})) = \mathcal{D}(\mathcal{S}_p(\mathcal{T})) = \mathcal{D}(\mathcal{T}),$$

whence the discrete tree $\mathcal{D}(\mathcal{T})$ is p-self-similar as well. For the converse direction, introduce some more notation:

- $\mathbb{T}$: The space of locally finite discrete rooted trees (endowed with topology of local convergence).
- $\mathbb{T}_e \subset \mathbb{T}$: The subspace of trees with a finite number of ends.
- $\iota : \mathbb{T} \to \mathfrak{T}^\ell$: embedding of a discrete tree into $\mathfrak{T}^\ell$ where each edge gets edge length 1 and μ = length measure.
- $\mathcal{M}_1(X)$ (for a metric space X): the space of probability measures on X, endowed with the Prokhorov topology.

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Let T be a p-self-similar random discrete tree, i.e. $T \overset{\text{law}}{=} \mathcal{C}_p(T)$. Then $T \in \mathbb{T}_e$ almost surely. We show

1. $\mathcal{D}(\mathcal{S}_{p^n}(\iota(T))) \to T$ as $n \to \infty$.
2. The sequence of laws of $\mathcal{S}_{p^n}(\iota(T))$ is precompact in $\mathcal{M}_1(\mathfrak{T}^\ell_e)$.
3. $\mathcal{D} : \mathcal{M}_1(\mathfrak{T}^\ell_e) \to \mathcal{M}_1(\mathbb{T}_e)$ is continuous and injective.

This implies the theorem.

Proof of Theorem S (3)

1. $\mathcal{D}(\mathcal{S}_{p^n}(\iota(T))) \to T$ as $n \to \infty$. Construct coupling between the trees T and $\mathcal{D}(\mathcal{S}_{p^n}(\iota(T)))$, or rather, suitably truncated versions of them

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A sequence of random trees $\mathcal{T}_1, \mathcal{T}_2, \ldots \in \mathcal{M}_1(\mathfrak{T}_e)$ is precompact in $\mathcal{M}_1(\mathfrak{T}_e)$ if and only if it is precompact in $\mathcal{M}_1(\mathfrak{T})$ and for every $r \ge 0$ there exist $R = R(r)$ and $n_0 = n_0(r)$, such that

the family of random variables $\quad (N_{r,R(r)}(\mathcal{T}_n))_{r\in\mathbb{N}, n\ge n_0(r)} \quad$ is tight.

Argue by contradiction using (technical) estimates.

Proof of Theorem S (3)

Last point to show: $\mathcal{D} : \mathcal{M}_1(\mathfrak{T}^\ell_e) \to \mathcal{M}_1(\mathbb{T}_e)$ is continuous and injective.
Through (non-trivial, but technical) truncation arguments, reduce to showing that $\mathcal{D}$ is continuous and injective on $\mathcal{M}_1(\mathfrak{T}^\ell_1)$. In fact, we have

Theorem T

The map $\mathcal{D}$ is a homeomorphism between $\mathcal{M}_1(\mathfrak{T}^\ell_e)$ and $\mathcal{D}(\mathcal{M}_1(\mathfrak{T}^\ell_e))$.

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In order to prove Theorem T, we will use two other representations of a random tree $\mathcal{T} \in \mathcal{M}_1(\mathfrak{T}_1^\ell)$:
Distance matrix: Let $\mathcal{T} = (\mathcal{V}, d, \rho, \mu)$ a random tree taking values in $\mathfrak{T}_1^\ell$. Let $X_0 = \rho$ and $X_1, X_2, \dots$ be iid according to μ. Then the law of

$$(D_\mathcal{T}(i,j))_{i,j\in\mathbb{N}} = (d(X_i, X_j))_{i,j\in\mathbb{N}},$$

also denoted by $D_\mathcal{T}$, is called the *distance matrix distribution* of the tree $\mathcal{T}$.

Theorem (Gromov, Greven–Pfaffelhuber–Winter)

The map $(\mathcal{T} \mapsto D_\mathcal{T})$ is a homeomorphism between $\mathcal{M}_1(\mathfrak{T}_1^\ell)$ and its image.

Proof of Theorem S (4)

Exchangeable partial order: Let $\mathcal{T} \in \mathcal{M}_1(\mathfrak{T}_1^\ell)$. Define a random partial order on $\{1, \dots, n\}$ as follows:

- Condition the tree $\mathcal{D}(\mathcal{T})$ on having n non-root vertices; label them uniformly at random by $1, \dots, n$.
- The *ancestral relation* of the resulting tree defines a random partial order on $\{1, \dots, n\}$.

By design, the sequence of random partial order thus obtained is compatible and thus extends to a random partial order $\lhd_{\mathcal{T}}$ on $\mathbb{N}$; this random partial order is moreover *exchangeable* by design.

Lemma

The maps $\mathcal{D}$ and $\lhd$: $\mathcal{T} \mapsto \lhd_{\mathcal{T}}$ induce the same topology on $\mathcal{M}_1(\mathfrak{T}_1^\ell)$.

Proof of Theorem T

Goal: Show that $\lhd$ is a homeomorphism between $\mathcal{M}_1(\mathfrak{T}_1^\ell)$ and its image.
Since $\mathcal{M}_1(\mathfrak{T}_1^\ell)$ is compact, enough to show that it is continuous and injective.
Injectivity of $\lhd$: Can reconstruct distance matrix $D_\mathcal{T}$ from $\lhd_\mathcal{T}$:

$$D_\mathcal{T}(i,j) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1, k\notin\{i,j\}}^{n} \mathbf{1}_{k\lhd_\mathcal{T} i,\, k\not\lhd_\mathcal{T} j \text{ or } k\lhd_\mathcal{T} j,\, k\not\lhd_\mathcal{T} i},$$

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Continuity of $\lhd$: Enough to show continuity on $\mathfrak{T}_1^\ell$ (deterministic trees). Show in fact that the map $\mathcal{T} \mapsto (D_\mathcal{T}, \lhd_\mathcal{T})$ is continuous on $\mathfrak{T}_1^\ell$. For this, consider expectation of test functions of the form

$$f(D,\lhd) = C \prod_{i,j=0}^{n} D(i,j)^{\beta_{ij}} \prod_{l=1}^{L} \mathbf{1}_{a_l \lhd b_l},$$

where $C\in\mathbb{R}$, $n\in\mathbb{N}$, $\beta_{ij}\in\mathbb{N}$, $L\geq 0$ and $a_l, b_l \in \{1,\dots,n\}$.

Proof of Theorem T (2)

$$f(D, \lhd) = C \prod_{i,j=0}^{n} D(i,j)^{\beta_{ij}} \prod_{l=1}^{L} \mathbf{1}_{a_l \lhd b_l},$$

where $C \in \mathbb{R}$, $n \in \mathbb{N}$, $\beta_{ij} \in \mathbb{N}$, $L \geq 0$ and $a_l, b_l \in \{1, \ldots, n\}$. Show that $\mathbb{E}[f(D_{\mathcal{T}}, \lhd_{\mathcal{T}})]$ is continuous in $\mathcal{T}$. Proof by induction over L.

$L = 0$: Follows from Gromov/Greven–Pfaffelhuber–Winter.

$L - 1 \to L$: Can assume that there exists $l_0 \in \{1, \ldots, L\}$ such that $a_{l_0} \notin \{b_l : l = 1, \ldots, L\}$ (otherwise there exists a cycle $a_{l_1} \lhd b_{l_1'} \lhd a_{l_2} \lhd \cdots \lhd b_{l_k'} \lhd a_{l_1}$ and thus $f \equiv 0$. Let Λ be the set of those $l \in \{1, \ldots, L\}$ for which the indicator $\mathbf{1}_{a_{l_0} \lhd b_l}$ appears in f. Then can prove that

$$\mathbb{E}\Big[\prod_{l \in \Lambda} \mathbf{1}_{a_{l_0} \lhd b_l} \,\Big|\, D, (\mathbf{1}_{a_l \lhd b_l}, l \notin \Lambda)\Big]$$

is a polynomial in $(D(i,j))_{i,j=1}^{n}$. This allows to complete the induction.

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- Proof of Theorem T is yet another example of the use of exchangeability in studying continuum limits of discrete structures.